Exam. Code : 103206
Subject Code : 1191

B.A./B.Sc. 6th Semester
MATHEMATICS
Paper—I (Linear Algebra)

Time Allowed—Three Hours] [Maximum Marks—50

Note :— Attempt **FIVE** questions in all selecting at least **TWO** questions from each section. All questions carry equal marks.

SECTION—A

1. (a) Prove that Z, the set of all integers, forms a group w.r.t. the binary operation $*$ defined as $a * b = a + b + 2 \ \forall \ a, b \in Z$.

   (b) If W_1 and W_2 are subspaces of V(F), then prove that $W_1 \cup W_2$ is a subspace of V(F) iff either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

2. (a) Let $x = (1, 2, -1)$; $y = (2, -3, 2)$; $z = (4, 1, 3)$; $w = (-3, 1, 2)$ be vectors in $\mathbb{R}^3(\mathbb{R})$. Show that $L(\{x, y\}) \neq L(\{z, w\})$.

   (b) Let V be a vector space over F, prove that for $v_1, v_2, v_3 \in V$:

   (i) the set $\{v_1, v_2\}$ is L.D. iff v_1 and v_2 are collinear.

   (ii) the set $\{v_1, v_2, v_3\}$ is L.D. iff v_1, v_2, v_3 are coplanar.

3. (a) Find k if the vectors $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix}$ are L.D.

(b) Prove that any linearly independent set in vector space V(F) can be extended to a basis of V.

4. (a) Let V be the vector space of all 2×2 symmetric matrices over $\mathbb{R}$. Find a basis and dimension of V.

(b) Let W be a subspace of finite dimensional vector-space V(F), prove that dim W $\leq$ dim V.

5. (a) In $\mathbb{R}^3$, let W_1, W_2 be the subspaces generated by {(1, 0, –1), (2, 1, 3)} and {(–1, 2, 2), (2, 2, –1), (3, 0, –3), (2, –1, 2)} respectively. Find the dimension of (i) W_1, (ii) W_2, (iii) $W_1 \cap W_2$, (iv) $W_1 + W_2$.

(b) If W is a subspace of a finite dimensional vector space V(F), prove that dim V/W = dim V – dim W.

SECTION—B

6. (a) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that T(1, 2) = (3, –1, 5) and T(0, 1) = (2, 1, –1).

(b) Show that a mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by T(x, y, z) = (x + 2, y, z) is not a linear transformation.

7. (a) If V(F) and W(F) are two vector spaces over the same field F and $T : V \to W$ be a L.T. Suppose V is finite dimensional. Prove that :

$$\text{Rank } T + \text{Nullity } T = \dim V.$$

(b) Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ whose range space is generated by (1, 2, 0, –4) and (2, 0, –1, –3).

8. (a) Prove that every n-dimensional vector space over the field F is isomorphic to F^n.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x - 2y - z, y - z, x)$. Prove that T is invertible and find T^{-1}.

9. (a) Let $B = \{v_1, v_2, ..., v_n\}$ be a basis for vector space V(F) and $T : V \to V$ be a L.T., prove that for any $v \in V$, $[T; B]\,[v; B] = [T(v); B]$.

(b) Find the matrix representation of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$T(x, y) = (x + 4y, 2x + 3y, 3x - 5y)$ w.r.t. ordered bases :

$B_1 = \{(1, 1), (2, 3)\}$ for $\mathbb{R}^2$ and

$B_2 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ for $\mathbb{R}^3$.

10. (a) Let $M = \begin{bmatrix} 3 & -4 & 5 \\ 1 & 2 & -6 \end{bmatrix}$ be the matrix which determines the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(V) = MV \ \forall\ V \in \mathbb{R}^3$. Find the matrix of T w.r.t. the bases $B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ for $\mathbb{R}^3$ and $B_2 = \{(1, 2), (3, 4)\}$ for $\mathbb{R}^2$.

(b) Let $B_1 = \{(1, 0), (0, 1)\}$ and $B_2 = \{(1, 3), (2, 5)\}$ be two ordered bases for $\mathbb{R}^2$. Find the transition matrix P from B_1 to B_2 and the transition matrix Q from B_2 to B_1. Also verify $Q = P^{-1}$ and $P[v; B_2] = [v; B_1]$.